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Estimates on moments of the solutions to stochastic differential equations with respect to martingales in the plane [1]

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Abstract

Let $M = \{M_z, z \in R^2_+\}$ be a two-parameter strong martingale, A be a two-parameter increasing process on $R^2_+ = [0, +\infty) \times [0, +\infty)$. Consider the following stochastic differential equations in the plane:

$$X_z = X_0 + \int_{R_z} a(\xi, X)\,\mathrm{d}M_\xi + \int_{R_z} b(\xi, X)\,\mathrm{d}A_\xi$$

for $z \in R^2_+$. Under some assumptions on the coefficients a, b and the integrators M, A, we prove the existence and uniqueness of solutions for the equations, and obtain some estimates on moments of solution.

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1. Introduction

Reid (1983) gave an estimate on moments of the solutions to the following stochastic differential equations:

$$X_z = \int_{R_z} e(\xi, X)\,\mathrm{d}W_\xi + \int_{R_z} f(\xi, X)\,\mathrm{d}\xi \tag{1.1}$$

under conditions that the coefficients e and f satisfy some growth conditions, where W is a two-parameter Wiener process on $R^2_+ = [0, +\infty) \times [0, +\infty)$. In this paper we are concerned with the two-parameter stochastic differential equations with respect

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to martingales,

$$X_z = X_0 + \int_{R_z} a(\xi,X)\,\mathrm{d}M_\xi + \int_{R_z} b(\xi,X)\,\mathrm{d}A_\xi \tag{1.2}$$

for $z \in R_+^2$, where M is a two-parameter strong martingale, and A is a two-parameter increasing process on R_+^2. We will study the existence and uniqueness of solutions for Eq. (1.2) in Section 3, and obtain our main results on the estimates of solutions in Section 4. Reid's (1983, Theorem 4.2) result can be considered as a special case of Theorem 4.1. Our results are based on two-parameter Ito's formula and Gronwall's inequality.

2. Preliminaries

Let $R_+^2 = [0,+\infty) \times [0,+\infty)$, ∂R_+^2 be its boundary. If $z = (s,t)$ and $z' = (s',t')$ are two points in R_+^2, we write $z \leqslant z'$ *iff* $s \leqslant s'$ and $t \leqslant t'$, $z < z'$ *iff* $s < s'$ and $t < t'$, $z \wedge z'$ *iff* $s < s'$ and $t > t'$. We shall adopt the notations: $z \otimes z' = (s,t')$, $z \vee z' = (\max(s,s'), \max(t,t'))$, $z \overline{\wedge} z' = (\min(s,s'), \min(t,t'))$. We also use the notation $(z,z'] = \{\xi \in R_+^2, z < \xi \leqslant z'\}$ for the left open rectangle, when $z < z'$. For a fixed $z \in R_+^2$, R_z will denote the rectangle $[0,z] = \{\xi \in R_+^2, \xi \leqslant z\}$. If f is a map from R_+^2 to R, then the increment of f on the rectangle $(z_1,z_2]$ is given by $f((z_1,z_2]) = f(z_2) - f(z_1 \otimes z_2) - f(z_2 \otimes z_1) + f(z_1)$.

Let $(\boldsymbol{\Omega}, \mathscr{F}, \boldsymbol{P})$ be a complete probability space, and let $\{\mathscr{F}_z, z \in R_+^2\}$ be a family of sub σ-fields of $\mathscr{F}$ satisfying the usual axioms as introduced in Cairoli and Walsh (1975). A two-parameter stochastic process $M = \{M_z,\ z \in R_+^2\}$ is said to be a strong martingale if (1) M is adapted; (2) M vanishes on the axes; (3) $\boldsymbol{E}\{M((z,z'])|\mathscr{F}_z^1 \vee \mathscr{F}_z^2\} = 0$, whenever $z < z'$, and if we replace (3) by (3)$'$ $\boldsymbol{E}\{M((z,z'])|\mathscr{F}_z\} = 0$, whenever $z < z'$, then M is said to be a weak martingale, where $\mathscr{F}_z^1 \equiv \sigma(\bigcup_{v\in[0,+\infty)} \mathscr{F}_{(s,v)})$ and $\mathscr{F}_z^2 \equiv \sigma(\bigcup_{u\in[0,+\infty)} \mathscr{F}_{(u,t)})$ for $z = (s,t)$.

We will also use the concepts of adapted 1- and 2-martingale as introduced in Cairoli and Walsh (1975). It is well known (cf. Cairoli and Walsh, 1975; Wong and Zokai, 1977) that a strong martingale is a martingale; a two-parameter stochastic process is a martingale if and only if it is both an adapted 1-martingale and an adapted 2-martingale; adapted 1- and 2-martingales are weak martingales. We know that (Merzbach and Zakai, 1980) for any square integrable martingale M there exists $\mathscr{F}_z^i$-predictable increasing process $[M]^i$ $(i = 1, 2)$ and $\mathscr{F}_z$-predictable increasing process $\langle M \rangle$ such that $(M)^2 - [M]^i$ $(i = 1, 2)$ is an $\bar{i}$-martingale ($\bar{i} = 1,2$, when $i = 2,1$, respectively) and $(M)^2 - \langle M \rangle$ is a weak martingale. Moreover, the concepts of the following types of stochastic integral $\int_{R_z} \phi_\xi\,\mathrm{d}M_\xi$, $\int\int_{R_z\times R_z} \psi(\xi,\eta)\,\mathrm{d}M_\xi\,\mathrm{d}M_\eta$ and $\int\int_{R_z\times R_z} \psi(\xi,\eta)\,\mathrm{d}\mu_\xi\,\mathrm{d}M_\eta$ introduced by Wong and Zakai (1977) will be used.

In order to obtain our main results, we prepare some basic facts.

Theorem 2.1 (Two-parameter Ito's formula). *Let $a(\omega,z,x)$ and $b(\omega,z,x)$ be real functions on $\boldsymbol{\Omega} \times R_+^2 \times R$, M be a right continuous square integrable martingale such that $\langle M \rangle = [M]^1 = [M]^2$ and A be a continuous adapted increasing process with $M_z =$*

$A_z = 0$ for $z \in \partial R_+^2$, and suppose that there exists a real stochastic process $X = \{X_z, z \in R_+^2\}$ such that the two-parameter stochastic process $\{\int_{R_z} a(\xi,X)\,\mathrm{d}M_\xi,\ z \in R_+^2\}$ is a continuous square integrable martingale, and $\{\int_{R_z} b(\xi,X)\,\mathrm{d}A_\xi,\ z \in R_+^2\}$ is a continuous adapted process with bounded variation. If we define a stochastic process Y by

$$Y_z = Y_0 + \int_{R_z} a(\xi,X)\,\mathrm{d}M_\xi + \int_{R_z} b(\xi,X)\,\mathrm{d}A_\xi,$$

then for any real function $f \in C^4(R)$ we have

$$\begin{aligned}
&f(Y_z) - f(Y_0)\\
&= \int_{R_z} f'(Y_\xi)a(\xi,X)\mathrm{d}M_\xi + \int_{R_z} f'(Y_\xi)b(\xi,X)\mathrm{d}A_\xi\\
&\quad + \frac{1}{2}\int_{R_z} f''(Y_\xi)a^2(\xi,X)\,\mathrm{d}\langle M\rangle_\xi\\
&\quad + \int\int_{R_z\times R_z} I(\eta\wedge\xi)f''(Y_{\xi\otimes\eta})a(\eta,X)a(\xi,X)\,\mathrm{d}M_\eta\,\mathrm{d}M_\xi\\
&\quad + \int\int_{R_z\times R_z} I(\eta\wedge\xi)f''(Y_{\xi\otimes\eta})b(\eta,X)b(\xi,X)\,\mathrm{d}A_\eta\,\mathrm{d}A_\xi\\
&\quad + \frac{1}{4}\int\int_{R_z\times R_z} I(\eta\wedge\xi)f^{(4)}(Y_{\xi\otimes\eta})a^2(\eta,X)a^2(\xi,X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}\langle M\rangle_\xi\\
&\quad + \int\int_{R_z\times R_z} I(\eta\wedge\xi)f''(Y_{\xi\otimes\eta})b(\eta,X)a(\xi,X)\,\mathrm{d}A_\eta\,\mathrm{d}M_\xi\\
&\quad + \int\int_{R_z\times R_z} I(\eta\wedge\xi)f''(Y_{\xi\otimes\eta})a(\eta,X)b(\xi,X)\,\mathrm{d}M_\eta\,\mathrm{d}A_\xi\\
&\quad + \frac{1}{2}\int\int_{R_z\times R_z} I(\eta\wedge\xi)f^{(3)}(Y_{\xi\otimes\eta})a^2(\eta,X)a(\xi,X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}M_\xi\\
&\quad + \frac{1}{2}\int\int_{R_z\times R_z} I(\eta\wedge\xi)f^{(3)}(Y_{\xi\otimes\eta})a(\eta,X)a^2(\xi,X)\,\mathrm{d}M_\eta\,\mathrm{d}\langle M\rangle_\xi\\
&\quad + \frac{1}{2}\int\int_{R_z\times R_z} I(\eta\wedge\xi)f^{(3)}(Y_{\xi\otimes\eta})a^2(\eta,X)b(\xi,X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}A_\xi\\
&\quad + \frac{1}{2}\int\int_{R_z\times R_z} I(\eta\wedge\xi)f^{(3)}(Y_{\xi\otimes\eta})b(\eta,X)a^2(\xi,X)\,\mathrm{d}A_\eta\,\mathrm{d}\langle M\rangle_\xi,
\end{aligned} \tag{2.1}$$

where

$$I(\eta\wedge\xi) = \begin{cases} 1 & \textit{if } \eta\wedge\xi,\\ 0 & \textit{otherwise}. \end{cases}$$

Proof. For a fixed $z = (s_0, t_0) \in R_+^2$, we know that $\{Y_{(s,t_0)}, s \in R_+\}$ is a semimartingale w.r.t. $(\mathscr{F}_z^1)_{z\in R_+^2}$ and $\{Y_{(s_0,t)}, t \in R_+\}$ is a semimartingale w.r.t. $(\mathscr{F}_z^2)_{z\in R_+^2}$. By using one-parameter Ito's formula (see Ikeda and Watanabe, Theorem 5.1) and a procedure similar to that of Proposition 5.1 in Wong and Zakai (1978) we will complete the proof. □

Theorem 2.2 (see Chevalier, 1982; B-D-G's inequality). *For any two-parameter right continuous square integrable martingale M and real number $p > 1$, there exists $C_p > 0$ and $\widetilde{C}_p > 0$ such that*

$$E(M_z^{*p}) \leqslant C_p E(\langle M\rangle_z^{p/2}) \leqslant \widetilde{C}_p E(M_z^{*p})$$

hold for $z \in R_+^2$, where $M_z^* = \sup_{z' \leqslant z} |M_{z'}|$.

Theorem 2.3 (Gronwall's inequality). *Assume that f_z, g_z and h_z are non-negative non-decreasing functions on R_+^2, and B_z is an increasing function on R_+^2, with $B_z = 0$ for $z \in \partial R_+^2$ satisfying*

$$f_z \leqslant g_z + h_z \left[\int_{R_z} f_\xi \, \mathrm{d}B_\xi + \int_0^s f_{\sigma,t} \, \mathrm{d}_\sigma B_{\sigma t} + \int_0^t f_{s,\tau} \, \mathrm{d}_\tau B_{s\tau} \right]$$

for $z = (s,t) \in R_+^2$. Then $f_z \leqslant g_z \exp\{3h_z B_z\}$.

Proof. By induction on n, for every integer $n \geqslant 1$, it is easy to show that

$$f_z \leqslant g_z \sum_{k=0}^{n-1} \frac{[3h_z B_z]^k}{k!} + f_z \frac{[3h_z B_z]^n}{n!}.$$

Letting $n \to +\infty$ in the last inequality, we have

$$f_z \leqslant g_z \sum_{n=0}^{\infty} \frac{[3h_z B_z]^n}{n!} = g_z \exp\{3h_z B_z\}. \quad \square$$

3. Existence and uniqueness theorem

Let $\mathbf{C} = \mathbf{C}(R_+^2)$ be the space of all continuous real-valued functions on R_+^2, $\mathscr{B}(\mathbf{C})$ be the σ-field on $\mathbf{C}$, $\mathscr{B}(R_+^2)$ be the Borel σ-field on R_+^2. Let $\mathscr{B}_z(\mathbf{C})$ be the σ-field generated by the cylinder sets $\{\omega \in \mathbf{C}; \omega(\xi) \in E\}$, where $E \in \mathscr{B}(R)$ and $\xi \leqslant z$.

We write $\mathbf{M}(\mathbf{\Omega} \times R_+^2 \times \mathbf{C})$ for the collection of all real-valued function α on $\mathbf{\Omega} \times R_+^2 \times \mathbf{C}$ satisfying the following measurability conditions:

1^0 α is an $\mathscr{F} \times \mathscr{B}(R_+^2) \times \mathscr{B}(\mathbf{C})/\mathscr{B}(R)$ measurable transformation of $\mathbf{\Omega} \times R_+^2 \times \mathbf{C}$ into R;

2^0 for every $z \in R_+^2$, $\alpha(\cdot, z, \cdot)$ is an $\mathscr{F}_z \times \mathscr{B}_z(\mathbf{C})/\mathscr{B}(R)$ measurable transformation of $\mathbf{\Omega} \times \mathbf{C}$ into R.

We write χ_{c} for the collection of all continuous adapted two-parameter stochastic process on R_+^2. For $X \in \chi_{\mathrm{c}}$, z_0, $z \in R_+^2$, we write

$$X_z^* = \sup_{\xi \leqslant z} |X_\xi|, \quad \|X\|_z = \{E\{X_z^{*2}\}\}^{1/2}, \quad \|X\| = \sum_{k=0}^{\infty} 2^{-k} \min\{\|X\|_{(k,k)}, 1\},$$

$$S(R_+^2) = \{X : X \in \chi_{\mathrm{c}}, \ \|X\|_z < +\infty, \ \text{for } z \in R_+^2\},$$

$$X^{z_0} = \{X_{z \curlywedge z_0}\}_{z \in R_+^2}, \quad S(z_0) = \{X^{z_0} : X \in S(R_+^2)\}.$$

We have (Yeh, 1981) that $S(R_+^2)$ is a Banach space equipped with the norm $\|\cdot\|$ and $S(z_0)$ is its subspace. Let Γ_c denote the collection of continuous adapted stochastic process on $\mathbf{\Omega}\times\partial R_+^2$. Similarly, we define Z_z^*, $\|Z\|_z$, $\|Z\|$ for any $Z \in \Gamma_c$ and $\mathbf{S}(\partial R_+^2) = \{Z : Z \in \Gamma_c,\ \|Z\|_z < +\infty \quad \text{for } z \in \partial R_+^2\}$.

In this section, we consider the following stochastic differential equations:

$$X_z = Z_{(s,0)} + Z_{(0,t)} - Z_{(0,0)} + \int_{R_z} a(\xi, X)\,\mathrm{d}M_\xi + \int_{R_z} b(\xi, X)\,\mathrm{d}A_\xi \tag{3.1}$$

for $z = (s,t) \in R_+^2$, where $Z \in \Gamma_c$, A is a continuous increasing process with $A_z = 0$ for $z \in \partial R_+^2$, M is a continuous square integrable martingale with $M_z = 0$ for $z \in \partial R_+^2$, and a, $b \in \mathbf{M}(\mathbf{\Omega}\times R_+^2 \times \mathbf{C})$. We write it as $X_z \equiv Z_{(s,0)} + Z_{(0,t)} - Z_{(0,0)} + a(X)\cdot M_z + b(X)\cdot A_z$ for short.

Furthermore, we suppose that there exists a non-negative predictable process $L = \{L_z | z \in R_+^2\}$ and an increasing function B on R_+^2 such that

(C.1) For every $\omega \in \mathbf{\Omega}$ and $G(z,\omega) \equiv \int_{R_z}[1 + L_\xi(\omega)]\,\mathrm{d}[A_\xi(\omega) + \langle M\rangle_\xi(\omega)]$, the random measure generated by $G_z(\omega)$ is dominated by the measure generated by B, that is,

$$\mathrm{d}G(\cdot,\omega) \leqslant \mathrm{d}B(\cdot) \text{ a.s.} \tag{3.2}$$

(C.2) For every $z \in R_+^2$, x, $y \in \mathbf{C}$, we have

$$\begin{aligned} &|a(\omega,z,x) - a(\omega,z,y)|^2 \leqslant L_z(\omega)(x-y)_z^{*2},\\ &|b(\omega,z,x) - b(\omega,z,y)| \leqslant L_z(\omega)(x-y)_z^{*}. \end{aligned} \tag{3.3}$$

(C.3) For every $z \in R_+^2$, $x \in \mathbf{C}$, we have

$$\begin{aligned} &|a(\omega,z,x)|^2 \leqslant L_z(\omega)(1 + x_z^{*2}),\\ &|b(\omega,z,x)| \leqslant L_z(\omega)(1 + x_z^{*}). \end{aligned} \tag{3.4}$$

(C.4) For every fixed ω, x, $a(\omega,z,x)$ is left continuous in z.

Proposition 3.1. *If the coefficients a, b and the integrators M, A in Eq.* (3.1) *satisfy the conditions* (C.1) (C.3) (C.4), *then the stochastic integral $\int_{R_z} a(\xi,X)\,\mathrm{d}M_\xi (\equiv a(X)\cdot M_z)$ is well defined for any $X \in \chi_c$.*

Proof. Since, for any $X \in \chi_c$ and $z \in R_+^2$, the transformation $\omega| \longrightarrow (\omega, X(\omega,\cdot))$: $\mathbf{\Omega} \to (\mathbf{\Omega} \times \mathbf{C})$ is $\mathscr{F}_z/\mathscr{F}_z \times \mathscr{B}_z(\mathbf{C})$ measurable and $a(\cdot,z,\cdot)$ is $\mathscr{F}_z \times \mathscr{B}_z(\mathbf{C})/\mathscr{B}(R)$ measurable, then by (C.4) their composition transformation $a(\cdot,z,X(\cdot))$ is $\mathscr{F}_z/\mathscr{B}(R)$ measurable, and $a(\cdot,z,X(\cdot))$ is a predictable process. Let

$$T_{N,z}(\omega) = \begin{cases} 1 & X_z^*(\omega) \leqslant N,\\ 0 & \text{otherwise.} \end{cases}$$

$N = 1,2,\ldots$. By Wong and Zakai (1976), we know that $\{T_N\}$ are $\{\mathscr{F}_z, z \in R_+^2\}$-predictable stopping times and $T_N \uparrow 1$ (as $N \to +\infty$). By (C.1) and (C.3), we get

$$E\left\{\int_{R_z} a^2(\xi,X)T_{N,\xi}\,\mathrm{d}\langle M\rangle_\xi\right\} \leqslant E\left\{\int_{R_z}(1+X_\xi^{*2})T_{N,\xi}L_\xi\,\mathrm{d}\langle M\rangle_\xi\right\}$$
$$\leqslant (1+N^2)B_z < +\infty.$$

So we can define the stochastic integral $\widetilde{Y}_z^N = \int_{R_z} a(\xi,X)T_{N,\xi}\,\mathrm{d}M_\xi$ and it is easy to see that

$$\widetilde{Y}^{(N+1)}T_N = \widetilde{Y}^N T_N,$$
$$\langle \widetilde{Y}^N\rangle_z = \int_{R_z} a^2(\xi,X)T_{N,\xi}\,\mathrm{d}\langle M\rangle_\xi \;\uparrow\; \int_{R_z} a^2(\xi,X)\,\mathrm{d}\langle M\rangle_\xi$$

(as $N \to +\infty$), which implies that there exists a stochastic process $\widetilde{Y}$ such that $\widetilde{Y}_z = \lim_{N\to+\infty}\widetilde{Y}_z^N$, a.s. $\omega \in \mathbf{\Omega}$. Hence, we can define the $\widetilde{Y}$ as stochastic integral of a w.r.t. M, and we denote it by $a(X)\cdot M_z$. It is easy to see that for $z \leqslant z'$,

$$a(X^{z'})\cdot M_z = a(X)\cdot M_z. \qquad \square \tag{3.5}$$

Lemma 3.1. *Let τ be a mapping from $\mathbf{S}(R_+^2)$ to $\mathbf{S}(R_+^2)$ satisfying the following conditions: there exists an increasing function $\overline{B}$ and a non-negative non-decreasing function f_z on R_+^2 such that*

1^0 For all X, $X' \in \mathbf{S}(R_+^2)$ and $z \in R_+^2$,

$$\|\tau X\|_z^2 \leqslant f_z + f_z\int_{R_z}\|X\|_\xi^2\,\mathrm{d}\overline{B}_\xi, \tag{3.6}$$

$$\|\tau X - \tau X'\|_z^2 \leqslant f_z\int_{R_z}\|X - X'\|_\xi^2\,\mathrm{d}\overline{B}_\xi. \tag{3.7}$$

2^0 For every $X' \in \mathbf{S}(R_+^2)$,

$$(\tau X^{z'})_z = (\tau X)_z \;,\; z \leqslant z'. \tag{3.8}$$

Then there exists a unique element $X \in \mathbf{S}(R_+^2)$ such that

$$\tau X = X. \tag{3.9}$$

Proof. For a fixed $z_0 \in R_+^2$, X, $X' \in \mathbf{S}(R_+^2)$, by induction on integer $k \geqslant 1$ and using (3.6)–(3.8) we get

$$\|\tau^k X\|_z^2 \leqslant f_z\sum_{i=0}^{k-1}\frac{(f_z\overline{B}_z)^i}{i!} + \frac{(f_z\overline{B}_z)^k}{k!}\|X\|_z^2 < +\infty, \tag{3.10}$$

$$\|\tau^k X - \tau^k X'\|_z^2 \leqslant \frac{(f_z\overline{B}_z)^k}{k!}\|X - X'\|_z^2 \tag{3.11}$$

for $z \leqslant z_0$. Choosing k_0 such that $(f_{z_0}\overline{B}_{z_0})^{k_0}/k_0! < 1$, we deduce from (3.10) and (3.11) that τ^{k_0} is a contraction from $\mathbf{S}(z_0)$ to $\mathbf{S}(z_0)$, hence, by fixed point theorem, there exists a unique element $X \in \mathbf{S}(z_0)$ such that $(\tau X)_z = X_z$ for $z \leqslant z_0$.

On the other hand, we can take an increasing sequence $\{z_n\} \subseteq R_+^2$ such that $\sum_{n=1}^{+\infty} [0, z_n] = R_+^2$. Also for any $n \geq 1$ we know from above that there exists $X^n \in \mathbf{S}(z_n)$ such that

$$(\tau X^n)_z = X_z^n, \quad z \leqslant z_n \tag{3.12}$$

which, together with (3.8), implies that for $z \leqslant z_n$

$$(\tau X^{(n+1)z_n})_z = (\tau X^{(n+1)})_z = X_z^{(n+1)} = X_z^{(n+1)z_n}.$$

So $X^{(n+1)z_n}$ is also a solution of Eq. (3.9), therefore, we have that $X_z^{(n+1)z_n} = X_z^{(n)}, z \leqslant z_n$. Let $X = \sum_{n=1}^{\infty} X^n I_{[0,\ z_n] \setminus [0,\ z_{n-1}]}$, then $X \in \mathbf{S}(R_+^2)$ is the unique solution of Eq. (3.9). □

Theorem 3.1. *Suppose that the coefficients a, b and the integrators M, A in Eq.* (3.1) *satisfy the conditions* (C.1)–(C.3). *Then for any* $Z \in \mathbf{S}(\partial R_+^2)$ *there exists a unique solution* $X \in \mathbf{S}(R_+^2)$ *of Eq.* (3.1).

Proof. Following an idea of Nie (1987), we give a sketch of proof for the sake of completeness. We define an operator τ from $\mathbf{S}(R_+^2)$ to $\mathbf{S}(R_+^2)$ by

$$(\tau X)_z = Z_{(s,0)} + Z_{(0,t)} - Z_{(0,0)} + \int_{R_z} a(\xi, X)\,\mathrm{d}M_\xi + \int_{R_z} b(\xi, X)\,\mathrm{d}A_\xi \tag{3.13}$$

for $z \in R_+^2$. We need only to show that for any $Z \in \mathbf{S}(\partial R_+^2)$, τ satisfies the conditions 1^0 and 2^0 of Lemma 3.1. By using Schwarz inequality, Theorem 2.2 and the conditions (C.1)–(C.3), we have for any X, $X' \in \mathbf{S}(R_+^2)$

$$\|a(X)\cdot M\|_z^2 \leqslant 16B_z + 16\int_{R_z} \|X\|_\xi^2\,\mathrm{d}B_\xi,$$

$$\|b(X)\cdot A\|_z^2 \leqslant 2B_z^2 + 2B_z \int_{R_z} \|X\|_\xi^2\,\mathrm{d}B_\xi,$$

$$\|(b(X) - b(X'))\cdot A\|_z^2 \leqslant B_z \int_{R_z} \|X - X'\|_\xi^2\,\mathrm{d}B_\xi,$$

$$\|(a(X) - a(X')\cdot M\|_z^2 \leqslant 16 \int_{R_z} \|X - X'\|_\xi^2\,\mathrm{d}B_\xi.$$

These inequalities together with (3.13) imply that

$$\|\tau X\|_z^2 \leqslant 5(3\|Z\|_z^2 + 2B_z^2 + 16B_z) + 10(B_z + 8)\int_{R_z} \|X\|_\xi^2\,\mathrm{d}B_\xi,$$

$$\|\tau X - \tau X'\|_z^2 \leqslant 2(B_z + 16)\int_{R_z} \|X - X'\|_\xi^2\,\mathrm{d}B_\xi.$$

Taking $f_z = \max\{5(3\|Z\|_z^2 + 2B_z^2 + 16B_z), 10(B_z + 8)\}$, we know from the last two inequalities that τ satisfies the condition 1^0 of Lemma 3.1. On the other hand, we see from (3.5) and the definition of τ that τ also satisfies the condition 2^0 of Lemma 3.1, and therefore the proof is completed. □

4. Main results

In this section, we consider Eq. (1.2), where M is a right continuous two-parameter square integrable strong martingale and A is a continuous $\{\mathscr{F}_z\}_{z\in R_+^2}$-adapted increasing process. We have the following.

Theorem 4.1. *Let X be a solution to Eq.* (1.2) *with coefficients a, b and integrators M, A satisfying conditions* (C.1), (C.3), (C.4). *in Section 3 and that $\langle M\rangle = [M]^1 = [M]^2$. Then for $z \in R_+^2$ and real number $n > 0$ there exists a constant C depending only on n such that the following inequalities hold*

(i) $EX_z^{*n} \leqslant (1+2E|X_0|^2)\exp\{3C(1+B_z)B_z\}, \quad 0 < n \leqslant 2,$

(ii) $EX_z^{*n} \leqslant (1+2E|X_0|^4)\exp\{72C(1+6B_z+2B_z^2)B_z^2\}, \quad 2 < n \leqslant 4,$

(iii) $EX_z^{*n} \leqslant (1+2E|X_0|^8)\exp\{72C(1+6B_z+2B_z^2)B_z^2\}, \quad 4 < n \leqslant 8,$

(iv) $EX_z^{*n} \leqslant (1+2E|X_0|^n)\exp\{72C(1+6B_z+2B_z^2)B_z^2\} - 1, \quad n > 8.$

Proof. Let

$$I_{N,\,z}(\omega) = \begin{cases} 1 & \text{when } X_z^*(\omega) \leqslant N, \\ 0 & \text{otherwise} \end{cases}$$

for $z \in R_+^2$ and $N = 1,2,\ldots$. Then as we see in the proof of Proposition 3.2, $I_N(\cdot)$ is a predictable process. We write

$$\begin{aligned}
&\hat{a}(X)_z \equiv \hat{a}(\omega,z,X) = a(\omega,z,X)I_{N,z}(\omega),\\
&\hat{b}(X)_z \equiv \hat{b}(\omega,z,X) = b(\omega,z,X)I_{N,z}(\omega),\\
&\hat{A}_z \equiv \hat{b}(X)\cdot A_z = \int_{R_z} \hat{b}(\xi,X)\,\mathrm{d}A_\xi,\\
&\hat{M}_z \equiv \hat{a}(X)\cdot M_z = \int_{R_z} \hat{a}(\xi,X)\,\mathrm{d}M_\xi,\\
&\hat{X}_0 = X_0 I_{N,(0,0)}(\omega), \quad \hat{X}_z = \hat{X}_0 + \hat{M}_z + \hat{A}_z.
\end{aligned} \tag{4.1}$$

By using conditions (C.1) and (C.3) we get that

$$\begin{aligned}
\mathrm{d}|\hat{A}_z| &\leqslant |b(z,X)|I_{N,z}\,\mathrm{d}A_z \leqslant L_z(1+X_z^*)I_{N,z}\,\mathrm{d}A_z\\
&\leqslant (1+X_z^*)I_{N,z}\,\mathrm{d}B_z \leqslant (1+N)\,\mathrm{d}B_z,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\mathrm{d}\langle\hat{M}\rangle_z &= a^2(z,X)I_{N,z}\,\mathrm{d}\langle M\rangle_z\\
&\leqslant L_z(1+X_z^{*2})I_{N,z}\,\mathrm{d}\langle M\rangle_z\\
&\leqslant (1+X_z^{*2})I_{N,z}\,\mathrm{d}B_z \leqslant (1+N^2)\,\mathrm{d}B_z.
\end{aligned} \tag{4.3}$$

Therefore, we have from (4.2), (4.3) and Theorem 2.2 that

$$E\{|\hat{M}_z|^p\} \leqslant C_p E\{\langle\hat{M}\rangle_z^{p/2}\} \leqslant C_p(1+N^2)^{p/2}B_z < +\infty,$$

$$E\{|\hat{A}_z|^p\} \leqslant (1+N^2)^{p/2}B_z < +\infty$$

for $p \geqslant 1$. Hence, $\widehat{a}$ and $\widehat{b}$ satisfy the conditions of Theorem 2.1. Let $f(x) = |x|^p$ ($p = 2$ or $p \geqslant 4$) for any $x \in R$; then $f(x) \in C^4(R)$ and by Theorem 2.1 we get

$$
\begin{aligned}
&\left| |\widehat{X}_z|^p - |\widehat{X}_0| \right|^p \\
&\leqslant \sup_{z' \leqslant z} \left| \int_{R_{z'}} f'(\widehat{X}_\xi)\widehat{a}(\xi, X)\,\mathrm{d}M_\xi \right| + \sup_{z' \leqslant z} \left| \int_{R_{z'}} f'(\widehat{X}_\xi)\widehat{b}(\xi, X)\,\mathrm{d}A_\xi \right| \\
&\quad + \frac{1}{2} \sup_{z' \leqslant z} \left| \int_{R_{z'}} f''(\widehat{X}_\xi)\widehat{a}^2(\xi, X)\,\mathrm{d}\langle M\rangle_\xi \right| \\
&\quad + \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f''(\widehat{X}_{\xi \otimes \eta})\widehat{a}(\eta, X)\widehat{a}(\xi, X)\,\mathrm{d}M_\eta\,\mathrm{d}M_\xi \right| \\
&\quad + \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f''(\widehat{X}_{\xi \otimes \eta})\widehat{b}(\eta, X)\widehat{b}(\xi, X)\,\mathrm{d}A_\eta\,\mathrm{d}A_\xi \right| \\
&\quad + \frac{1}{4} \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f^{(4)}(\widehat{X}_{\xi \otimes \eta})\widehat{a}^2(\eta, X)\widehat{a}^2(\xi, X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}\langle M\rangle_\xi \right| \\
&\quad + \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f''(\widehat{X}_{\xi \otimes \eta})\widehat{b}(\eta, X)\widehat{a}(\xi, X)\,\mathrm{d}A_\eta\,\mathrm{d}M_\xi \right| \\
&\quad + \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f''(\widehat{X}_{\xi \otimes \eta})\widehat{a}(\eta, X)\widehat{b}(\xi, X)\,\mathrm{d}M_\eta\,\mathrm{d}A_\xi \right| \\
&\quad + \frac{1}{2} \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f^{(3)}(\widehat{X}_{\xi \otimes \eta})\widehat{a}^2(\eta, X)\widehat{a}(\xi, X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}M_\xi \right| \\
&\quad + \frac{1}{2} \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f^{(3)}(\widehat{X}_{\xi \otimes \eta})\widehat{a}(\eta, X)\widehat{a}^2(\xi, X)\,\mathrm{d}M_\eta\,\mathrm{d}\langle M\rangle_\xi \right| \\
&\quad + \frac{1}{2} \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f^{(3)}(\widehat{X}_{\xi \otimes \eta})\widehat{a}^2(\eta, X)\widehat{b}(\xi, X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}A_\xi \right| \\
&\quad + \frac{1}{2} \sup_{z' \leqslant z} \left| \int\int_{R_{z'} \times R_{z'}} I(\eta \wedge \xi) f^{(3)}(\widehat{X}_{\xi \otimes \eta})\widehat{b}(\eta, X)\widehat{a}^2(\xi, X)\,\mathrm{d}A_\eta\,\mathrm{d}\langle M\rangle_\xi \right| \\
&\equiv I_1 + I_2 + I_3 + \cdots + I_{12}.
\end{aligned}
$$

So we have $\widehat{X}_z^{*p} - |\widehat{X}_0|^p \leqslant \sum_{i=1}^{12} I_i$ which implies that

$$
\begin{aligned}
\boldsymbol{E}(X_z^{*2p}) &\leqslant 2\boldsymbol{E}[(\widehat{X}_z^{*p} - |\widehat{X}_0|^p)^2 + |\widehat{X}_0|^{2p}] \\
&\leqslant 2\boldsymbol{E}|\widehat{X}_0|^{2p} + 24 \sum_{i=0}^{12} \boldsymbol{E}I_i^2.
\end{aligned} \tag{4.4}
$$

Now we estimate $\boldsymbol{E}I_1^2, \ldots, \boldsymbol{E}I_{12}^2$. For this we will use the following elemental inequality:

$$|\theta|^\alpha \leqslant 1 + |\theta|^{\alpha+\beta} \tag{4.5}$$

for $\alpha > 0$ and $\beta > 0$. In the sequel, all the constants depending only on p will be simply denoted by C; it may change from one inequality to another one. By (4.5), (C.1), (C.3) and Theorem 2.2 we get

$$\begin{aligned}
\boldsymbol{E}I_1^2 &= \boldsymbol{E}\left\{\sup_{z'\leqslant z}\left|\int_{R_{z'}} f'(\widehat{X}_\xi)\widehat{a}(\xi,X)\,\mathrm{d}M_\xi\right|\right\}^2 \\
&\leqslant C\boldsymbol{E}\left\{\int_{R_z} f'(\widehat{X}_\xi)^2\widehat{a}^2(\xi,X)\,\mathrm{d}\langle M\rangle_\xi\right\} \\
&\leqslant C\boldsymbol{E}\left\{\int_{R_z} [p\widehat{X}_\xi^{p-1} sign\{\widehat{X}_\xi\}]^2(1+\widehat{X}_\xi^{*2})I_{N,\xi}\,\mathrm{d}B_\xi\right\} \\
&\leqslant C\boldsymbol{E}\left\{\int_{R_z} |\widehat{X}_\xi^*|^{2p-2}(1+\widehat{X}_\xi^{*2})\,\mathrm{d}B_\xi\right\} \\
&\leqslant C\int_{R_z}(1+\boldsymbol{E}\widehat{X}_\xi^{*2p})\,\mathrm{d}B_\xi.
\end{aligned} \tag{4.6}$$

By (4.5), (C.1), (C.3) and Schwarz inequality we also get

$$\begin{aligned}
\boldsymbol{E}I_2^2 &= \boldsymbol{E}\left\{\sup_{z'\leqslant z}\left|\int_{R_{z'}} f'(\widehat{X}_\xi)\widehat{b}(\xi,X)\,\mathrm{d}A_\xi\right|\right\}^2 \\
&\leqslant \boldsymbol{E}\left\{\int_{R_z} |f'(\widehat{X}_\xi)||\widehat{b}(\xi,X)|\,\mathrm{d}A_\xi\right\}^2 \\
&\leqslant CB_z\boldsymbol{E}\left\{\int_{R_z} |\widehat{X}_\xi^*|^{2p-2}(1+\widehat{X}_\xi^*)^2 I_{N,\xi}\,\mathrm{d}B_\xi\right\} \\
&\leqslant CB_z\int_{R_z}(1+\boldsymbol{E}\widehat{X}_\xi^{*2p})\,\mathrm{d}B_\xi.
\end{aligned} \tag{4.7}$$

Similarly,

$$\boldsymbol{E}I_3^2 \leqslant CB_z\int_{R_z}(1+\boldsymbol{E}\widehat{X}_\xi^{*2p})\,\mathrm{d}B_\xi. \tag{4.8}$$

Noting $\langle M\rangle = [M]^1 = [M]^2$ we have from Proposition 2.4 of Cairoli and Walsh (1975) that

$$\left\langle \int\int_{R.\times R.} \Psi(\xi,\eta)\,\mathrm{d}M_\xi\,\mathrm{d}M_\eta \right\rangle_z = \int\int_{R_z\times R_z} \Psi^2(\xi,\eta)\,\mathrm{d}\langle M\rangle_\xi\,\mathrm{d}\langle M\rangle_\eta. \tag{4.9}$$

So we deduce from (4.5), (4.9), (C.1), (C.3) and Theorem 2.2 that

$$\begin{aligned}
\boldsymbol{E}I_4^2 &\leqslant C\boldsymbol{E}\left\{\int\int_{R_z\times R_z} I(\eta\wedge\xi)f''(\widehat{X}_{\xi\otimes\eta})^2\widehat{a}^2(\eta,X)\widehat{a}^2(\xi,X)\,\mathrm{d}\langle M\rangle_\eta\,\mathrm{d}\langle M\rangle_\xi\right\} \\
&\leqslant C\boldsymbol{E}\left\{\int\int_{R_z\times R_z} I(\eta\wedge\xi)\widehat{X}_{\eta\vee\xi}^{*2p-4}(1+X_\eta^{*2})(1+X_\xi^{*2})I_{N,\eta}I_{N,\xi}\,\mathrm{d}B_\eta\,\mathrm{d}B_\xi\right\}
\end{aligned}$$

$$\leqslant CE\left\{\int\int_{R_z\times R_z} I(\eta\wedge\xi)\widehat{X}^{*2p-4}_{\xi\vee\eta}(1+\widehat{X}^{*2}_{\xi\vee\eta})^2\,\mathrm{d}B_\xi\,\mathrm{d}B_\eta\right\}$$

$$= CE\left\{\int_{R_z}\widehat{X}^{*2p-4}_{\sigma\tau}(1+\widehat{X}^{*2}_{\sigma\tau})^2\,\mathrm{d}_\sigma B_{\sigma\tau}\,\mathrm{d}_\tau B_{\sigma\tau}\right\}$$

$$\leqslant CE\left\{\int_0^s(1+\widehat{X}^{*p}_{\sigma t})\,\mathrm{d}_\sigma B_{\sigma t}\int_0^t(1+\widehat{X}^{*p}_{s\tau})\,\mathrm{d}_\tau B_{s\tau}\right\}$$

$$\leqslant C\left[\boldsymbol{E}\left(\int_0^s(1+\widehat{X}^{*p}_{\sigma t})\,\mathrm{d}_\sigma B_{\sigma t}\right)^2+\boldsymbol{E}\left(\int_0^t(1+\widehat{X}^{*p}_{s\tau})\,\mathrm{d}_\tau B_{s\tau}\right)^2\right]$$

$$\leqslant CB_z\left[\int_0^s(1+\boldsymbol{E}\widehat{X}^{*2p}_{\sigma t})\,\mathrm{d}_\sigma B_{\sigma t}+\int_0^t(1+\boldsymbol{E}\widehat{X}^{*2p}_{s\tau})\,\mathrm{d}_\tau B_{s\tau}\right]. \tag{4.10}$$

Similarly, we have

$$\boldsymbol{E}I_5^2\leqslant CB_z^3\left[\int_0^s(1+\boldsymbol{E}\widehat{X}^{*2p}_{\sigma t})\,\mathrm{d}_\sigma B_{\sigma t}+\int_0^t(1+\boldsymbol{E}\widehat{X}^{*2p}_{s\tau})\,\mathrm{d}_\tau B_{s\tau}\right], \tag{4.11}$$

$$\boldsymbol{E}I_6^2\leqslant CB_z^3\left[\int_0^s(1+\boldsymbol{E}\widehat{X}^{*2p}_{\sigma t})\,\mathrm{d}_\sigma B_{\sigma t}+\int_0^t(1+\boldsymbol{E}\widehat{X}^{*2p}_{s\tau})\,\mathrm{d}_\tau B_{s\tau}\right]. \tag{4.12}$$

In order to estimate $\boldsymbol{E}I_7^2$, we need the following.

Lemma 4.1. *Assume that M is a continuous two-parameter strong martingale, A is an adapted continuous two-parameter stochastic process of finite variation and $\mathrm{d}|A_z|\leqslant\theta_z\,\mathrm{d}B_z$, where θ_z is a non-negative bounded, adapted continuous process. Let $\Psi(\xi,\eta)$ be a predictable process (see Cairoli and Walsh, 1975) satisfying*

(i) $\Psi(\xi,\eta)=0$ *unless* $\xi\wedge\eta$,

(ii) *for* $z_0\in R_+^2$, $\boldsymbol{E}\{\int\int_{R_{z_0}\times R_{z_0}}\Psi^2(\xi,\eta)\mathrm{d}|A_\xi|\,\mathrm{d}\langle M\rangle_\eta\}<+\infty$.

Let $\overline{Y}_z=\int\int_{R_z\times R_z}\Psi(\xi,\eta)\,\mathrm{d}A_\xi\,\mathrm{d}M_\eta$, *for* $z=(s,t)\leqslant z_0=(s_0,t_0)$; *then we have*

$$\boldsymbol{E}\{\overline{Y}^{*2}_{z_0}\}\leqslant 4\boldsymbol{E}\{\mathrm{var}_{t\leqslant t_0}\overline{Y}_{s_0t}\}^2\leqslant 4B_{z_0}\boldsymbol{E}\left\{\int\int_{R_{z_0}\times R_{z_0}}\Psi^2(\xi,\eta)\theta_\xi^2\,\mathrm{d}B_\xi\,\mathrm{d}\langle M\rangle_\eta\right\}.$$

Proof. The first inequality is obtained by Theorems 2.5 and 3.1 of Wong and Zakai (1976), therefore it suffices to show the second inequality. We divide the proof into two steps.

(i) $\Psi(\xi,\eta)=\alpha I_{D\times E}(\xi,\eta)$, where $D=(z_1,z_2]$ and $E=(\widetilde{z}_1,\widetilde{z}_2]$ are left open rectangles such that if $\eta\in D$ and $\eta'\in E$ then $\eta\wedge\eta'$, and α is an $\mathscr{F}_{z_1\vee\widetilde{z}_1}$-measurable bounded random variable. It is easy to prove that

$$\overline{Y}_z=\alpha A(D\cap R_z)M(E\cap R_z),$$

$$\mathrm{var}_{t\leqslant t_0}(\overline{Y}_{s_0t})\leqslant|A(D)||\alpha M(E)|.$$

(ii) $\Psi(\xi,\eta)=\sum_{i,j}\Psi_{i,j}I_{D_i\times E_j}(\xi,\eta)$, where $D_i=(z_i,z_i']$ and $E_j=(\widetilde{z}_j,\widetilde{z}_j']$ are left open rectangles such that if $\xi\in D_i$ and $\eta\in E_j$ then $\xi\wedge\eta$, and $\Psi_{i,j}$ is a $\mathscr{F}_{z_i\vee\widetilde{z}_j}$-measurable bounded random variable ($i, j = 1,2,\ldots,n.$); in this case we have

$$
\begin{aligned}
\boldsymbol{E}(\mathrm{var}_{t\leqslant t_0}\overline{Y}_{s_0t})^2 &\leqslant \boldsymbol{E}\left\{\sum_i |A|(D_i)|\sum_j \Psi_{i,j}M(E_j)|\right\}^2 \\
&\leqslant \boldsymbol{E}\left\{\sum_i \left(\int_{D_i}\theta_\xi\,\mathrm{d}B_\xi\right)\left|\sum_j \Psi_{i,j}M(E_j)\right|\right\}^2 \\
&\leqslant \boldsymbol{E}\left\{\sum_i B(D_i)^{1/2}\left(\int_{D_i}\theta_\xi^2\,\mathrm{d}B_\xi\right)^{1/2}\left|\sum_j \Psi_{i,j}M(E_j)\right|\right\}^2 \\
&\leqslant \boldsymbol{E}\left\{\sum_i B(D_i)\sum_i\int_{D_i}\theta_\xi^2\,\mathrm{d}B_\xi\left(\sum_j \Psi_{i,j}M(E_j)\right)^2\right\} \\
&\leqslant B_{z_0}\boldsymbol{E}\left\{\sum_i\int_{D_i}\theta_\xi^2\,\mathrm{d}B_\xi\left(\sum_j \Psi_{i,}M(E_j)\right)^2\right\} \\
&\qquad \text{(by the strong martingale property of } M) \\
&\leqslant B_{z_0}\boldsymbol{E}\left\{\sum_i\int_{D_i}\theta_\xi^2\,\mathrm{d}B_\xi\sum_j \Psi_{i,j}^2\langle M\rangle(E_j)\right\} \\
&= B_{z_0}\boldsymbol{E}\left\{\sum_{i,j}\Psi_{i,j}^2\int_{D_i}\theta_\xi^2\,\mathrm{d}B_\xi\langle M\rangle(E_j)\right\} \\
&= B_{z_0}\boldsymbol{E}\left\{\int\int_{R_z\times R_z}\Psi^2(\xi,\eta)\theta_\xi^2\,\mathrm{d}B_\xi\,\mathrm{d}\langle M\rangle_\eta\right\}.
\end{aligned}
$$

By taking *limit* in the last inequality, we complete the proof. □

Now we turn to estimate $\boldsymbol{E}I_7^2$. By Lemma 4.1, similar to the proof of (4.10), and noting $\mathrm{d}|\widehat{b}(X)\cdot A_\eta|\leqslant(1+X_\eta^*)I_{N,\eta}\,\mathrm{d}B_\eta$ we have

$$
\begin{aligned}
\boldsymbol{E}I_7^2 &= \boldsymbol{E}\left\{\sup_{z'\leqslant z}\left|\int\int_{R_{z'}\times R_{z'}} I(\eta\wedge\xi)f''(\widehat{X}_{\xi\otimes\eta})\widehat{b}(\eta,X)\widehat{a}(\xi,X)\,\mathrm{d}A_\eta\,\mathrm{d}M_\xi\right|\right\}^2 \\
&\leqslant 4B_z\boldsymbol{E}\left\{\int\int_{R_z\times R_z} I(\eta\wedge\xi)f''(\widehat{X}_{\xi\otimes\eta})^2\widehat{a}^2(\xi,X)(1+X_\eta^*)^2 I_{N,\eta}\,\mathrm{d}B_\eta\,\mathrm{d}\langle M\rangle_\xi\right\} \\
&\leqslant CB_z\boldsymbol{E}\left\{\int\int_{R_z\times R_z} I(\eta\wedge\xi)\widehat{X}_{\eta\vee\xi}^{*2p-4}(1+\widehat{X}_{\eta\vee\xi}^*)^2(1+\widehat{X}_{\eta\vee\xi}^{*2})\,\mathrm{d}B_\eta\,\mathrm{d}B_\xi\right\} \\
&\leqslant CB_z\boldsymbol{E}\left\{\int\int_{R_z\times R_z} I(\eta\wedge\xi)\widehat{X}_{\eta\vee\xi}^{*2p-4}(1+\widehat{X}_{\eta\vee\xi}^{*2})^2\,\mathrm{d}B_\eta\,\mathrm{d}B_\xi\right\} \\
&= CB_z\boldsymbol{E}\left\{\int_{R_z} |\widehat{X}_{\sigma\tau}^*|^{2p-4}(1+\widehat{X}_{\sigma\tau}^{*2})^2\,\mathrm{d}_\sigma B_{\sigma\tau}\,\mathrm{d}_\tau B_{\sigma\tau}\right\} \\
&\leqslant CB_z^2\left[\int_0^s(1+\boldsymbol{E}\widehat{X}_{\sigma t}^{*2p})\,\mathrm{d}_\sigma B_{\sigma t}+\int_0^t(1+\boldsymbol{E}\widehat{X}_{s\tau}^{*2p})\,\mathrm{d}_\tau B_{s\tau}\right].
\end{aligned}
\tag{4.13}
$$

In the same way as for EI_7^2 we have

$$EI_i^2 \leqslant CB_z^2\left[\int_0^s (1+E\widehat{X}_{\sigma t}^{*2p})\,\mathrm{d}_\sigma B_{\sigma t} + \int_0^t (1+E\widehat{X}_{s\tau}^{*2p})\,\mathrm{d}_\tau B_{s\tau}\right] \tag{4.14}$$

($i = 8, 9, 10, 11, 12$). Let $f_z = 1 + E\widehat{X}_z^{*2p}$, $g_z = 1 + 2E|\widehat{X}_0|^{2p}$, $h_z = 24C(1 + 6B_z + 2B_z^2)B_z$ for $z \in R_+^2$. Then by substituting (4.6)–(4.8), (4.10)–(4.14) into (4.4) we get

$$f_z \leqslant g_z + h_z\left[\int_{R_z} f_\xi\,\mathrm{d}B_\xi + \int_0^s f_{\sigma t}\,\mathrm{d}_\sigma B_{\sigma t} + \int_0^t f_{s\tau}\,\mathrm{d}_\tau B_{s\tau}\right].$$

Hence, by Theorem 2.3 we have

$$E\widehat{X}_z^{*2p} \leqslant (1 + 2E|\widehat{X}_0|^{2p})\exp\{72C(1 + 6B_z + 2B_z^2)B_z^2\} - 1.$$

Letting $N \to +\infty$ in the last inequality we have

$$EX_z^{*2p} \leqslant (1 + 2E|X_0|^{2p})\exp\{72C(1 + 6B_z + 2B_z^2)B_z^2\} - 1, \tag{4.15}$$

where $p = 2$ or $p \geqslant 4$, and the constant C depends only on p.

Now, by using (4.15) we shall give estimates on EX_z^{*n} for every real number $n \geqslant 0$.

(a) For $0 < n \leqslant 2$, we have from (4.1) that

$$\begin{aligned} E\widehat{X}_z^{*2} &\leqslant 2E|\widehat{X}_0|^2 + 2E|\widehat{A}_z^*|^2 + E|\widehat{M}_z^*|^2 \\ &\leqslant 2E|\widehat{X}_0|^2 + CE\left\{\int_{R_z}(1+\widehat{X}_\xi^*)\,\mathrm{d}B_\xi\right\}^2 + CE\left\{\int_{R_z}(1+\widehat{X}_\xi^{*2})\,\mathrm{d}B_\xi\right\} \\ &\leqslant 2E|\widehat{X}_0|^2 + C(1+B_z)\int_{R_z}(1+E\widehat{X}_\xi^{*2})\,\mathrm{d}B_\xi. \end{aligned}$$

Similar to the proof of (4.15), we also have

$$EX_z^{*2} \leqslant (1 + 2E|X_0|^2)\exp\{3C(1 + B_z)B_z\} - 1 \tag{4.16}$$

for $z \in R_+^2$. Therefore, by (4.5) and (4.16) we get

$$EX_z^{*n} \leqslant 1 + EX_z^{*2} \leqslant (1 + 2E|X_0|^2)\exp\{3C(1 + B_z)B_z\}. \tag{4.17}$$

(b) For $2 < n \leqslant 4$, by (4.5) and (4.15) we get

$$EX_z^{*n} \leqslant 1 + EX_z^{*4} \leqslant (1 + 2E|X_0|^4)\exp\{72C(1 + 6B_z + 2B_z^2)B_z^2\}. \tag{4.18}$$

(c) For $4 < n \leqslant 8$, by (4.5) and (4.15) we get

$$EX_z^{*n} \leqslant 1 + EX_z^{*8} \leqslant (1 + 2E|X_0|^8)\exp\{72C(1 + 6B_z + 2B_z^2)B_z^2\}. \tag{4.19}$$

(d) For $n > 8$, by (4.15) we get

$$EX_z^{*n} \leqslant (1 + 2E|X_0|^n)\exp\{72C(1 + 6B_z + 2B_z^2)B_z^2\} - 1. \tag{4.20}$$

A combination of (4.17)–(4.20) implies Theorem 4.1. □

Remark 4.1. In the special case that L is a constant, $M \equiv W$ is the Wiener process on R_+^2 and $A_{st} = s \cdot t$ in (C.1)–(C.3), Theorem 4.1 is reduced to J. Reid's results (see Reid, 1983, Theorem 4.2).

Remark 4.2. Let the predictable process L be a constant, $M = \int_{R_z} f(\xi) dW_\xi$, where f is a bounded predictable process, $A_{st} = s \cdot t$ in (C.1)–(C.3); then M is a strong martingale and $\langle M \rangle(z) = [M]^1(z) = [M]^2(z) = \int_{R_z} f(\xi)^2 d\xi$. It is easy to see that the L, M and A satisfy the conditions (C.1)–(C.4). In this case we can take B in (C.1) as $B_{(st)} = s \cdot t$.

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